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Differences of Alexander polynomials for knots caused by a single crossing change

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ABSTRACT

Kondo and Sakai independently gave a characterization of Alexander polynomials for knots which are transformed into the trivial knot by a single crossing change. The first author gave a characterization of Alexander polynomials for knots which are transformed into the trefoil knot (and into the figure-eight knot) by a single crossing change. In this note, we will give a characterization of Alexander polynomials for knots which are transformed into the 10_{132} knot (and into the $(5,2)$-torus knot) by a single crossing change. Moreover, this method can be applied for knots with monic Alexander polynomials.

1. Introduction

A *knot* is a simple closed oriented curve in the three-dimensional sphere S^3. For a knot k, an *Alexander matrix* $M_k(t)$ of k is a presentation matrix of the first integral homology group $H_1(\widetilde{X_\infty})$ as a Λ-module, where $\widetilde{X_\infty}$ means the infinite cyclic covering space of the exterior X of k in S^3 and Λ means the integral group-ring $\mathbf{Z}H_1(X)$; we can see that $\mathbf{Z}H_1(X) = \Lambda$ is the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ where t is always taken to be represented by the meridian of k. The Λ-module $H_1(\widetilde{X_\infty})$ is said to be the *Alexander invariant* (or Alexander module). An *Alexander polynomial* $\Delta_k(t)$ of k is a generator of the order ideal of $M_k(t)$. Throughout this note, we assume that $\Delta_k(t) = a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i})$. Therefore, the Alexander polynomial of k is determined up to signs.

In 1937, Wendt [8] introduced a notion of operation for links. We usually call the operation an *unknotting operation* (or briefly, a *crossing change*), which is defined to be a local move between two knot diagrams K_1 and K_2 which are identical except near one point as in Fig. 1. Furthermore, we consider its spatial realization as follows: For two knots k_1 and k_2 represented by K_1 and K_2, k_1 and k_2 are said to be transformed into each other by a single crossing change. If a knot k is transformed to the trivial knot by a single crossing change, k is said to be a *knot with unknotting number one*.

In this note, we will give an approach to characterize Alexander polynomials for knots which are transformed into a given knot by a single crossing change (Proposition 6). We will actuary give a characterization for the cases of the 10_{132} knot and the $(5,2)$-torus knot 5_1 as in Fig. 2. In Section 2, we consider a surgery description of knots to give an approach of characterization. This method can be applied for knots with monic Alexander polynomials.

Let k be a knot, $k^\times$ the set of all knots obtained from k by a single crossing change. Let $\Delta k^\times$ be the set of Alexander polynomials for knots in $k^\times$.

Theorem 1. *Let $F(t)$ be a Laurent polynomial. $F(t)$ belongs to $\Delta 10_{132}^\times$ if and only if $F(t)$ satisfies the following three conditions*:

(1) $F(t) = F(t^{-1})$.

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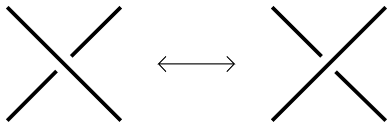

Fig. 1. Crossing change.

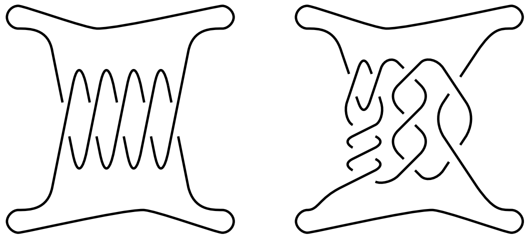

Fig. 2. 5_1 and 10_{132}.

(2) $F(1) = \pm 1$.
(3) $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, *where* $\alpha = a^2 + b^2 + c^2 + d^2 - ab - 2ac + 2ad - bc - 2bd - cd$, *and* $\beta = -2(a^2 + b^2 + c^2 + d^2) + ab + 3ac - 3ad + bc + 3bd + cd$ *for some integers a, b, c, and d.*

The proof of Theorem 1 is given in Section 3.

Theorem 2. *Let $F(t)$ be a Laurent polynomial. $F(t)$ belongs to $\Delta 5_1^{\times}$ if and only if $F(t)$ satisfies the following three conditions*:

(1) $F(t) = F(t^{-1})$.
(2) $F(1) = \pm 1$.
(3) $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, *where* $\alpha = 2(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2) - ab - 3ac + 3ad - 3ae + 3af + ag - 4ah - bc - 3bd - be - 3bf + 3bg + bh - cd + 4ce - cf - 3cg + 3ch - de + 4df - dg - 3dh - ef - 3eg + 3eh - fg - 3fh - gh$, *and* $\beta = -3(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2) + 2ab + 5ac - 5ad + 5ae - 5af - 2ag + 6ah + 2bc + 5bd + 2be + 5bf - 5bg - 2bh + 2cd - 6ce + 2cf + 5cg - 5ch + 2de - 6df + 2dg + 5dh + 2ef + 5eg - 5eh + 2fg + 5fh + 2gh$ *for some integers a, b, c, d, e, f, g, and h.*

The proof of Theorem 2 is given in Section 4.

We remark that $\Delta_{10_{132}}(t) = \Delta_{5_1}(t) = t^2 - t + 1 - t^{-1} + t^{-2}$, and furthermore that their Alexander invariants are identical. But the sets of Alexander polynomials of their neighborhoods, $\Delta 10_{132}^{\times}$ and $\Delta 5_1^{\times}$, are different as follows. Here, $\Delta\mathcal{K}$ is the set of Alexander polynomials for all knots.

Corollary 3. (1) $\Delta 10_{132}^{\times} \cap \Delta 5_1^{\times} \neq \emptyset$.
(2) $\Delta 10_{132}^{\times} \setminus \Delta 5_1^{\times} \neq \emptyset$.
(3) $\Delta 5_1^{\times} \setminus \Delta 10_{132}^{\times} \neq \emptyset$.
(4) $\Delta\mathcal{K} \setminus (\Delta 10_{132}^{\times} \cup \Delta 5_1^{\times}) \neq \emptyset$.

The proof of Corollary 3 is given in Section 5.

For the cases of knots 3_1, 4_1, 5_1, and 10_{132}, the sets of Alexander polynomials of their neighborhoods are proper subsets of $\Delta\mathcal{K}$. It is known that $\Delta k^{\times} = \Delta\mathcal{K}$ for a knot k with $\Delta_k(t) = 1$. We raise the following question.

Question. Does there exist a knot k with $\Delta_k(t) \neq 1$ such that $\Delta k^{\times} = \Delta\mathcal{K}$?

In Section 6, we study which prime knots with 10 crossings or less belong to $10_{132}^{\times}$ and to $5_1^{\times}$, and which Alexander polynomials of prime knots with 10 crossings or less belong to $\Delta 10_{132}^{\times}$ and to $\Delta 5_1^{\times}$.

2. Surgery description

It is well-known that any knot can be transformed into the trivial knot by crossing changes at suitable crossing points. Every crossing change is obtained by a ± 1 surgery along a small trivial knot around the crossing point with linking number 0. Levine [2] and Rolfsen [6,7] introduced a surgery description of a knot and a surgical view of Alexander matrix and Alexander polynomial as follows:

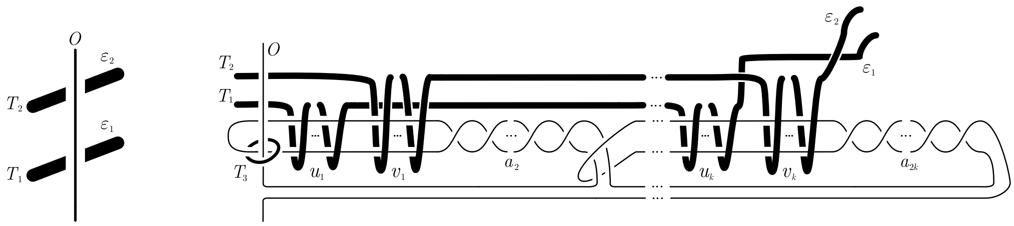

Fig. 3. Surgery description.

Proposition 4. *([6]) Let k be a knot, k_0 the trivial knot. Then, there exist n disjoint solid tori $T_1, \ldots, T_n$ in $S^3 \setminus k_0$ and a homeomorphism ϕ from $S^3 \setminus {}^\circ T_1 \cup \cdots \cup {}^\circ T_n$ to itself such that*

(1) $\phi(k_0) = k$,
(2) $T_1 \cup \cdots \cup T_n$ *is a trivial link,*
(3) $\mathrm{lk}(T_i, k_0) = \mathrm{lk}(T_i, k) = 0$ *for each i, and*
(4) $\phi(\partial T_i) = \partial T_i$ *and* $\mathrm{lk}(\mu_i', T_i) = 1$ *where* $\mu_i \subset \partial T_i$ *is a meridian of* T_i *and* $\mu_i' = \phi^{-1}(\mu_i)$.

From a surgery description, we have a surgical view of Alexander matrix of the knot as follows:

Proposition 5. *([6]) Let k be a knot and n the number in Proposition 4. Then, k has an Alexander matrix $M_k(t) = (m_{ij}(t))_{1 \leqslant i,j \leqslant n}$ satisfying (1) $m_{ij}(t) = m_{ji}(t^{-1})$, and (2) $|m_{ij}(1)| = \delta_{ij}$, where δ_{ij} is the Kronecker delta, that is, $\delta_{ij} = \begin{cases} 1 & (\text{if } i = j) \\ 0 & (\text{if } i \neq j) \end{cases}$.*

By a surgical view of Alexander matrices, we can have the following.

Proposition 6. *Let k be a knot and $M_k(t)$ the Alexander matrix in Proposition 5. Then a Laurent polynomial $F(t)$ belongs to $\Delta k^\times$ if and only if there exist Laurent polynomials $r_1(t), \ldots, r_n(t)$, and $m(t)$ such that*

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_i(1) = 0$ $(i = 1, \ldots, n)$, *and*
(2) $F(t) = \pm \det \begin{pmatrix} & & & r_1(t^{-1}) \\ & M_k(t) & & \vdots \\ & & & r_n(t^{-1}) \\ r_1(t) & \cdots & r_n(t) & m(t) \end{pmatrix}$.

Proof. In the case $n = 1$, the proof of this proposition was given in [4,5]. Here, we give a sketch of the proof in the case $n = 2$. It is clear the necessity. So, we give the sufficiency. Since $m(t) = m(t^{-1})$, $m(1) = \pm 1$, and $r_1(1) = r_2(1) = 0$, we rewrite

$$m(t) = \pm 1 + (a_2 + 1)\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 + \cdots + (-1)^k (a_{2k-2} + 1)\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^{2k-2} + (-1)^{k+1} a_{2k}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^{2k},$$

$$r_1(t) = \pm t^{l_1}\left(u_1(1-t) + u_2(1-t)^2 + \cdots + u_k(1-t)^n\right), \quad \text{and}$$

$$r_2(t) = \pm t^{l_2}\left(v_1(1-t) + v_2(1-t)^2 + \cdots + v_k(1-t)^n\right).$$

By the hypothesis $n = 2$, a surgery description of k is given by the trivial knot O and two solid tori, T_1 and T_2, as in the left of Fig. 3, where the solid tori are illustrated by thick lines. We transform this part of k into the right of Fig. 3 by a single crossing change, and we obtain the new knot $k_1 \in k^\times$. The single crossing change is obtained by $m(1)$ surgery along T_3, which is illustrated by a thin line. Here, $u_1, \ldots, u_k$; $v_1, \ldots, v_k$ mean the numbers of left-handed linkings of each part of solid tori and each parallel parts of the knot, and $a_2, a_4, \ldots, a_{2k}$ mean the numbers of left-handed full-twists of each parallel parts of the knot.

We recall that k_1 is obtained from k by a single crossing change. A crossing change is realized by a ± 1 surgery along T_3 around the crossing point with linking number 0. In Fig. 4, a part of the fundamental region of the infinite cyclic covering space of the exterior of k_1 is illustrated. By reading the linking numbers between lifts of T_1, T_2, and T_3, we can calculate $r_1(t), r_2(t)$, and $m(t)$.

Then, from a surgical viewpoint, k_1 has a surgical view of Alexander matrix of the following form:

$$\begin{pmatrix} m_{11}(t) & m_{12}(t) & r_1(t^{-1}) \\ m_{21}(t) & m_{22}(t) & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}.$$

In the case $n \geqslant 3$, we can prove similarly. The proof is complete. □

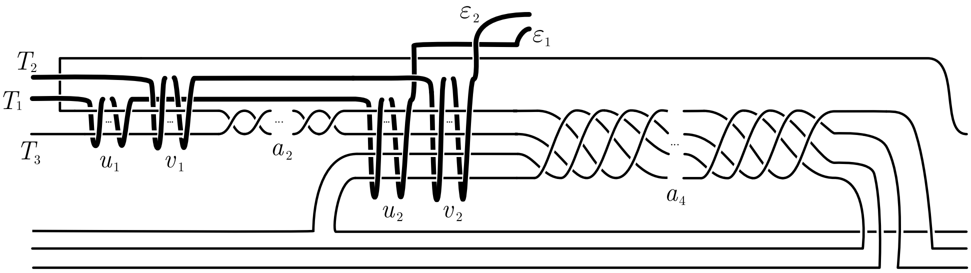

Fig. 4. Infinite cyclic cover.

3. Proof of Theorem 1

In order to show Theorem 1, we use the following two lemmas.

Lemma 7. *Let $F(t)$ be a Laurent polynomial. $F(t)$ belongs to $\triangle 10^{\times}_{132}$ if and only if there exist Laurent polynomials $m(t)$ and $r(t)$ such that*

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r(1) = 0$, *and*
(2) $F(t) = \pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$.

Since 10_{132} has a surgical view of Alexander matrix $(t^2 - t + 1 - t^{-1} + t^{-2})$, Lemma 7 follows Proposition 6.

Lemma 8. *Let $F(t)$ be a Laurent polynomial. There exist Laurent polynomials $m(t)$ and $r(t)$ such that*

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r(1) = 0$, *and*
(2) $F(t) = \pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$

if and only if $F(t)$ satisfies the following three conditions:

(1) $F(t) = F(t^{-1})$.
(2) $F(1) = \pm 1$.
(3) $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, *where* $\alpha = a^2 + b^2 + c^2 + d^2 - ab - 2ac + 2ad - bc - 2bd - cd$, *and* $\beta = -2(a^2 + b^2 + c^2 + d^2) + ab + 3ac - 3ad + bc + 3bd + cd$ *for some integers a, b, c, and d.*

Proof. Let $F(t) = \pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$.

Since $m(t)$ and $r(t)$ satisfy $m(t) = m(t^{-1})$, $m(1) = \pm 1$, $r(1) = 0$, we have (1) $F(t) = F(t^{-1})$, and (2) $F(1) = \pm 1$.

From now, we show (3). Since $r(1) = 0$, we rewrite $r(t) = t^l(t-1)(u_n t^n + u_{n-1}t^{n-1} + \cdots + u_1 t + u_0)$. Then, there exist four integers a, b, c, and d such that

$$\begin{aligned} r(t) &\equiv t^l(t-1)\big((u_{n-1} + u_n)t^{n-1} + (u_{n-2} - u_n)t^{n-2} + (u_{n-3} + u_n)t^{n-3} \\ &\quad + (u_{n-4} - u_n)t^{n-4} + u_{n-5}t^{n-5} + \cdots + u_1 t + u_0\big) \\ &\ \vdots \\ &\equiv t^l(t-1)\big(at^3 + bt^2 + ct + d\big) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}. \end{aligned}$$

Therefore, we have $F(t) = \pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix} \equiv \mp r(t)r(t^{-1}) \equiv \pm(t-1)(1-t^{-1})(at^3 + bt^2 + ct + d)(d + ct^{-1} + bt^{-2} + at^{-3}) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, where $\alpha = a^2 + b^2 + c^2 + d^2 - ab - 2ac + 2ad - bc - 2bd - cd$, $\beta = -2(a^2 + b^2 + c^2 + d^2) + ab + 3ac - 3ad + bc + 3bd + cd$.

On the other hand, for four integers a, b, c, and d, let $F(t)$ be a Laurent polynomial such that $F(1) = \varepsilon = \pm 1$, and $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, where α and β are as above.

The proof of sufficiency is divided into two cases: $\deg F(t) \leqslant 4$ and $\deg F(t) > 4$, where $\deg F(t)$ means the difference of the maximum degree and the minimum degree of $F(t)$.

(i) $\deg F(t) \leqslant 4$. By the hypothesis, there exists an integer N such that $F(t) = N(t^2 - t + 1 - t^{-1} + t^{-2}) \pm (\alpha t + \beta + \alpha t^{-1})$. Since $F(1) = \varepsilon$ $(\varepsilon = \pm 1)$, we have $N = \varepsilon \mp (2\alpha + \beta)$.

Let $m(t) = -adt^2 + (-ac - bd + ad)t + (\varepsilon + 2ac + 2bd) + (-ac - bd + ad)t^{-1} - adt^{-2}$, and $r(t) = (t-1)(at^3 + bt^2 + ct + d)$. Then, they satisfy $m(1) = \varepsilon$, $m(t) = m(t^{-1})$, and $r(1) = 0$.

Hence, we have $F(t) = (\varepsilon \mp (2\alpha + \beta))t^2 + (-\varepsilon \pm (3\alpha + \beta))t + (\varepsilon \mp 2\alpha) + (-\varepsilon \pm (3\alpha + \beta))t^{-1} + (\varepsilon \mp (2\alpha + \beta))t^{-2} = \pm((t^2 - t + 1 - t^{-1} + t^{-2})m(t) - r(t)r(t^{-1})) = \pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$.

(ii) $\deg F(t) > 4$. Let $G(t) = (\varepsilon \mp (2\alpha + \beta))t^2 + (-\varepsilon \pm (3\alpha + \beta))t + (\varepsilon \mp 2\alpha) + (-\varepsilon \pm (3\alpha + \beta))t^{-1} + (\varepsilon \mp (2\alpha + \beta))t^{-2}$.

It can be seen that $G(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \equiv F(t) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$.

Since $\deg G(t) \leqslant 4$, $G(1) = \varepsilon$, $G(t) = G(t^{-1})$, there exist Laurent polynomials $m(t)$ and $r(t)$ such that $m(1) = \varepsilon$, $m(t) = m(t^{-1})$, $r(1) = 0$, and $G(t) = \pm\det\begin{pmatrix} t^2-t+t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$. Let $H(t) = G(t) - F(t)$. $H(t)$ satisfies $H(1) = G(1) - F(1) = \varepsilon - \varepsilon = 0$, $H(t^{-1}) = G(t^{-1}) - F(t^{-1}) = G(t) - F(t) = H(t)$, and $H(t) \equiv 0 \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$.

Therefore, there exists a Laurent polynomial $h(t)$ such that $H(t) = \pm(t-1)(t^{-1}-1)(t^2 - t + 1 - t^{-1} + t^{-2})h(t)$ and $h(t) = h(t^{-1})$.

Let $\widehat{m}(t) = m(t) - (t-1)(t^{-1}-1)h(t)$. $\widehat{m}(t)$ satisfies $\widehat{m}(1) = m(1) = \varepsilon$ and $\widehat{m}(t^{-1}) = m(t^{-1}) - (t^{-1}-1)(t-1)h(t^{-1}) = m(t) - (t-1)(t^{-1}-1)h(t) = \widehat{m}(t)$.

Hence, we have $\pm\det\begin{pmatrix} t^2-t+1-t^{-1}+t^{-2} & r(t^{-1}) \\ r(t) & \widehat{m}(t) \end{pmatrix} = \pm((t^2 - t + 1 - t^{-1} + t^{-2})m(t) - (t-1)(t^{-1}-1)(t^2 - t + 1 - t^{-1} + t^{-2})h(t) - r(t)r(t^{-1})) = G(t) - H(t) = G(t) - (G(t) - F(t)) = F(t)$. The proof is complete. $\square$

4. Proof of Theorem 2

The proof of Theorem 2 is parallel to that of Theorem 1, except for the number of integers. In order to show Theorem 2, we use the following two lemmas.

Lemma 9. *Let $F(t)$ be a Laurent polynomial. $F(t)$ belongs to $\Delta 5_1^{\times}$ if and only if there exist Laurent polynomials $m(t), r_1(t)$, and $r_2(t)$ such that*

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_1(1) = r_2(t) = 0$, *and*
(2) $F(t) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}$.

Since 5_1 has a surgical view of Alexander matrix $\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 \\ -t+1 & -t+1-t^{-1} \end{pmatrix}$, Lemma 9 follows Proposition 6.

Lemma 10. *Let $F(t)$ be a Laurent polynomial. There exist Laurent polynomials $m(t)$, $r_1(t)$, and $r_2(t)$ such that*

(1) $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_1(1) = r_2(t) = 0$, *and*
(2) $F(t) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}$

if and only if $F(t)$ satisfies the following three conditions:

(1) $F(t) = F(t^{-1})$.
(2) $F(1) = \pm 1$.
(3) $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, *where* $\alpha = 2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) - ab - 3ac + 3ad - 3ae + 3af + ag - 4ah - bc - 3bd - be - 3bf + 3bg + bh - cd + 4ce - cf - 3cg + 3ch - de + 4df - dg - 3dh - ef - 3eg + 3eh - fg - 3fh - gh$, *and* $\beta = -3(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) + 2ab + 5ac - 5ad + 5ae - 5af - 2ag + 6ah + 2bc + 5bd + 2be + 5bf - 5bg - 2bh + 2cd - 6ce + 2cf + 5cg - 5ch + 2de - 6df + 2dg + 5dh + 2ef + 5eg - 5eh + 2fg + 5fh + 2gh$ *for some integers a, b, c, d, e, f, g, and h.*

Proof. Let $F(t) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}$.

Since $m(t)$ and $r(t)$ satisfy $m(t) = m(t^{-1})$, $m(1) = \pm 1$, and $r_1(1) = r_2(1) = 0$, we have (1) $F(t) = F(t^{-1})$, and (2) $F(1) = \pm 1$.

From now, we show (3). Since $r_1(1) = r_2(1) = 0$, we rewrite $r_1(t) = t^{l_1}(t-1)(u_n t^n + u_{n-1}t^{n-1} + \cdots + u_1 t + u_0)$, and $r_2(t) = t^{l_2}(t-1)(v_n t^n + v_{n-1}t^{n-1} + \cdots + v_1 t + v_0)$. Then, there exist some integers a, b, c, d, e, f, g, and h such that

$$r_1(t) \equiv (t-1)\big(at^3 + bt^2 + ct + d\big) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}, \quad \text{and}$$
$$r_2(t) \equiv (t-1)\big(et^3 + ft^2 + gt + h\big) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}.$$

Therefore, we have $F(t) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix} \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, where α and β are as above.

On the other hand, for eight integers a, b, c, d, e, f, g, and h, let $F(t)$ be a Laurent polynomial such that $F(1) = \varepsilon = \pm 1$, and $F(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, where α and β are as above.

The proof of sufficiency is divided into two cases: $\deg F(t) \leqslant 4$ and $\deg F(t) > 4$.

(i) $\deg F(t) \leqslant 4$. By the hypothesis, there exists an integer N such that $F(t) = N(t^2 - t + 1 - t^{-1} + t^{-2}) \pm (\alpha t + \beta + \alpha t^{-1})$. Since $F(1) = \varepsilon$ ($\varepsilon = \pm 1$), we have $N = \varepsilon \mp (2\alpha + \beta)$.

Let $m(t) = At^3 + Bt^2 + Ct + D + Ct^{-1} + Dt^{-2} + At^{-3}$, $r_1(t) = (t-1)(at^3 + bt^2 + ct + d)$, and $r_2(t) = (t-1)(et^3 + ft^2 + gt + h)$, where $A = ad - de + eh$, $B = ac - 2ad + ah + bd - ce + 2de - df + eg - 2eh + fh$, $C = ab - 2ac + ad + ag - 2ah + bc - 2bd - be + bh + cd + 2ce - cf - dg + 2df + ef - 2eg + eh + fg - 2fh + gh$, and $D = \varepsilon - 2ab + 2ac - 2ag + 2ah - 2bc + 2bd + 2be - 2bh - 2cd - 2ce + 2cf - 2de - 2df - 2ef + 2eg - 2fg + 2fh - 2gh$.

Then, they satisfy $m(1) = \varepsilon$, $m(t) = m(t^{-1})$, and $r_1(1) = r_2(1) = 0$.

Hence, we have $F(t) = (\varepsilon \mp (2\alpha + \beta))t^2 + (-\varepsilon \pm (3\alpha + \beta))t + (\varepsilon \mp 2\alpha) + (-\varepsilon \pm (3\alpha + \beta))t^{-1} + (\varepsilon \mp (2\alpha + \beta))t^{-2} = \pm((t^2 - t + 1 - t^{-1} + t^{-2})m(t) + (t - 1 + t^{-1})(r_1(t)r_1(t^{-1}) + r_2(t)r_2(t^{-1})) + (1 - t^{-1})r_1(t)r_2(t^{-1}) + (1 - t)r_1(t^{-1})r_2(t)) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}$.

(ii) $\deg F(t) > 4$. Let $G(t) = (\varepsilon \mp (2\alpha + \beta))t^2 + (-\varepsilon \pm (3\alpha + \beta))t + (\varepsilon \mp 2\alpha) + (-\varepsilon \pm (3\alpha + \beta))t^{-1} + (\varepsilon \mp (2\alpha + \beta))t^{-2}$.

It can be seen that $G(t) \equiv \pm(\alpha t + \beta + \alpha t^{-1}) \equiv F(t) \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$.

Since $\deg G(t) \leqslant 4$, $G(1) = \varepsilon$, $G(t) = G(t^{-1})$, there exist Laurent polynomials $m(t), r_1(t)$ and $r_2(t)$ such that $m(1) = \varepsilon$, $m(t) = m(t^{-1})$, $r_1(1) = r_2(1) = 0$, and $G(t) = \pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix}$. Let $H(t) = G(t) - F(t)$. $H(t)$ satisfies $H(1) = G(1) - F(1) = \varepsilon - \varepsilon = 0$, $H(t^{-1}) = G(t^{-1}) - F(t^{-1}) = G(t) - F(t) = H(t)$, and $H(t) \equiv 0 \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$.

Therefore, there exists a Laurent polynomial $h(t)$ such that $H(t) = \pm(t-1)(t^{-1}-1)(t^2 - t + 1 - t^{-1} + t^{-2})h(t)$ and $h(t) = h(t^{-1})$.

Let $\widehat{m}(t) = m(t) - (t-1)(t^{-1}-1)h(t)$. $\widehat{m}(t)$ satisfies $\widehat{m}(1) = m(1) = \varepsilon$ and $\widehat{m}(t^{-1}) = m(t^{-1}) - (t^{-1}-1)(t-1)h(t^{-1}) = m(t) - (t-1)(t^{-1}-1)h(t) = \widehat{m}(t)$.

Hence, we have

$$
\begin{aligned}
&\pm\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & \widehat{m}(t) \end{pmatrix} \\
&\quad = \pm\left(\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 & r_1(t^{-1}) \\ -t+1 & -t+1-t^{-1} & r_2(t^{-1}) \\ r_1(t) & r_2(t) & m(t) \end{pmatrix} - (t-1)\big(t^{-1}-1\big)h(t)\det\begin{pmatrix} -t+1-t^{-1} & -t^{-1}+1 \\ -t+1 & -t+1-t^{-1} \end{pmatrix}\right) \\
&\quad = G(t) - H(t) = G(t) - \big(G(t) - F(t)\big) = F(t).
\end{aligned}
$$

The proof is complete. □

Remark. The above arguments can be applied for knots with monic Alexander polynomials.

5. Proof of Corollary 3

Lemma 11. *For four integers a, b, c, and d, let $\alpha = a^2 + b^2 + c^2 + d^2 - ab - 2ac + 2ad - bc - 2bd - cd$, and $\beta = -2(a^2 + b^2 + c^2 + d^2) + ab + 3ac - 3ad + bc + 3bd + cd$. Then, we have*

(1) *$\beta \leqslant 0$. Moreover, $\beta = 0$ if and only if $a = b = c = d = 0$.*
(2) *$\frac{1+\sqrt{5}}{2}\beta \leqslant \alpha \leqslant \frac{1-\sqrt{5}}{2}\beta$.*
(3) *$3\alpha + \beta \equiv 0, \pm 1 \pmod 5$.*

Furthermore, for any n with $n \equiv 0, \pm 1 \pmod 5$, there exist four integers a, b, c, and d such that α and β satisfy $3\alpha + \beta = n$.

Proof. (1) $\beta = -2(b - \frac{a+c+3d}{4})^2 - \frac{15}{8}(c - \frac{13a+7d}{15})^2 - \frac{7}{15}(a + \frac{11}{14}d)^2 - \frac{5}{28}d^2 \leqslant 0$.

(2) $\alpha - \frac{1+\sqrt{5}}{2}\beta = (2+\sqrt{5})(b - \frac{(3+\sqrt{5})a + (3+\sqrt{5})c + (7+3\sqrt{5})d}{2(2+\sqrt{5})})^2 + \frac{65+29\sqrt{5}}{8(2+\sqrt{5})}(c - \frac{(65+29\sqrt{5})a - (32+26\sqrt{5})d}{2(65+29\sqrt{5})})^2 \geqslant 0$.

$\frac{1-\sqrt{5}}{2}\beta - \alpha = (-2+\sqrt{5})(b + \frac{(3-\sqrt{5})a + (3-\sqrt{5})c + (7-3\sqrt{5})d}{2(-2+\sqrt{5})})^2 + \frac{65-29\sqrt{5}}{8(-2+\sqrt{5})}(c + \frac{(65-29\sqrt{5})a + (32-26\sqrt{5})d}{2(-65+29\sqrt{5})})^2 \geqslant 0$.

(3) $3\alpha + \beta = a^2 + b^2 + c^2 + d^2 - 2ab - 3ac + 3ad - 2bc - 3bd - 2cd = 5(-ac + ad - bd) + (a - b + c - d)^2 \equiv 0, \pm 1 \pmod 5$.

If $3\alpha + \beta = 5k$, we take $(a, b, c, d) = (0, -k-1, -k, 1)$. If $3\alpha + \beta = 5k + 1$, we take $(a, b, c, d) = (0, -k, -k, 1)$. If $3\alpha + \beta = 5k - 1$, we take $(a, b, c, d) = (0, -k, 3-k, 1)$. The proof is complete. □

Lemma 12. *For eight integers a, b, c, d, e, f, g, and h, let* $\alpha = 2(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)-ab-3ac+3ad-3ae+3af+ag-4ah-bc-3bd-be-3bf+3bg+bh-cd+4ce-cf-3cg+3ch-de+4df-dg-3dh-ef-3eg+3eh-fg-3fh-gh$, *and* $\beta = -3(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)+2ab+5ac-5ad+5ae-5af-2ag+6ah+2bc+5bd+2be+5bf-5bg-2bh+2cd-6ce+2cf+5cg-5ch+2de-6df+2dg+5dh+2ef+5eg-5eh+2fg+5fh+2gh$. *Then, we have*

(1) $-2\alpha \leqslant \beta \leqslant 2\alpha$. *Moreover,* $-2\alpha = \beta$ *if and only if* $\alpha = \beta = 0$.
(2) $3\alpha + \beta \equiv 0, \pm 2 \pmod 5$.

Proof. (1) $\beta + 2\alpha = (b + \frac{-d-f+g}{2})^2 + (c + \frac{-a+2e-g+h}{2})^2 + \frac{3}{4}(a + \frac{2d+2f-g-3h}{3})^2 + \frac{5}{12}(d+f+g)^2 \geqslant 0$.

$2\alpha - \beta = 7(b + \frac{-4a-4c-11d-4e-11f+11g+4h}{14})^2 + \frac{45}{7}(c + \frac{-17a-10d+18e-10f-11g+17h}{18})^2 + \frac{25}{36}(a + \frac{4d+4f-g-5h}{5})^2 + \frac{1}{4}(d+f+g)^2 \geqslant 0$.

From this inequation, $-2\alpha = \beta$ implies that $A = 2b-d-f+g = 0$, $B = -a+2c+2e-g+h = 0$, $C = 3a+2d+2f-g-3h = 0$, and $D = d+f+g = 0$. Then, we have $\alpha = \frac{1}{2}A^2 + \frac{1}{2}B^2 + \frac{1}{9}C^2 + \frac{5}{36}D^2 - \frac{1}{4}AB - \frac{1}{4}AC - \frac{1}{6}BC - \frac{5}{12}BD + \frac{5}{36}CD + \frac{5}{36}gD = 0$, and $\alpha = \beta = 0$. The converse also holds.

(2) $3\alpha + \beta = 2(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)+ab-ac+ad+bc-bd+cd+ef-eg+eh+fg-fh+gh-ae+af-ag-4ah+be-bf+bg-bh+4ce+cf-cg+ch+de+4df+dg-dh \equiv 2(a-b+c-d+e-f+g-h)^2 \equiv 0, \pm 2 \pmod 5$. The proof is complete. □

Remark. Lemma 11(3) is also shown by the following. $3\alpha + \beta \equiv -2\alpha + \beta = \det\begin{pmatrix} -5 & r(-1) \\ r(-1) & m(-1) \end{pmatrix} \equiv -r(-1)^2 \equiv 0, \pm 1 \pmod 5$.

Lemma 12(2) is also shown by the following. $3\alpha + \beta \equiv -2\alpha + \beta = \det\begin{pmatrix} 3 & 2 & r_1(-1) \\ 2 & 3 & r_2(-1) \\ r_1(-1) & r_2(-1) & m(-1) \end{pmatrix} \equiv 2(r_1(-1) + r_2(-1))^2 \equiv 0, \pm 2 \pmod 5$.

Let k be a knot with unknotting number one. Then, k has a surgical view of Alexander matrix $M_k(t) = (\Delta_k(t))$. We have the following.

Proposition 13. *Let k be a knot with unknotting number one. If a Laurent polynomial* $F(t)$ *belongs to* $\Delta k^\times$, *then* $F(-1) \equiv -n^2 \mod \Delta_k(-1)$ *for some integer n.*

Remark. Let k be a knot with unknotting number one. Kawauchi [1] shows that a Laurent polynomial $F(t)$ belongs to $\Delta k^\times$ if and only if there exists a Laurent polynomial $r(t)$ such that $r(1) = 0$ and that $F(t) \equiv \pm r(t)r(t^{-1}) \pmod{\Delta_k(t)}$.

Proof of Corollary 3. We find a concrete Laurent polynomial $F(t)$ in each region. Take two integers p and q such that $F(t) \equiv pt + q + pt^{-1} \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$.

(1) We take $F(t) = t^2 - t + 1 - t^{-1} + t^{-2}$. Since $F(t) \equiv 0 \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, we have $p = 0$ and $q = 0$. By Lemma 11(1) (or Theorem 1), $a = b = c = d = 0$ imply $\alpha = 0$, $\beta = 0$, and $F(t) \in \Delta 10_{132}^\times$. By Theorem 2, $a = b = c = d = e = f = g = h = 0$ imply $\alpha = 0$, $\beta = 0$, and $F(t) \in \Delta 5_1^\times$.

(2) We take $F(t) = t^2 - 6t + 11 - 6t^{-1} + t^{-2}$. Since $F(t) \equiv -5t + 10 - 5t^{-1} \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, we have $p = -5$ and $q = 10$. By Theorem 1, $a = 1$, $b = 1$, $c = -1$, $d = -1$ imply $\alpha = 5$, $\beta = -10$, and $F(t) \in \Delta 10_{132}^\times$. There never exist α and β such that $-2p = q$ by Lemma 12(1). We have $F(t) \notin \Delta 5_1^\times$.

(3) We take $F(t) = 2t^2 - 6t + 7 - 6t^{-1} + 2t^{-2}$. Since $F(t) \equiv -4t + 5 - 4t^{-1} \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, we have $p = -4$ and $q = 5$. There never exist α and β such that $3\alpha + \beta = -7 \equiv -2 \pmod 5$, $3(-\alpha) - \beta = 7 \equiv 2 \pmod 5$ by Lemma 11(3). We have $F(t) \notin \Delta 10_{132}^\times$. By Theorem 2, $a = -3$, $b = -2$, $c = -3$, $d = -1$, $e = 3$, $f = 1$, $g = 1$, $h = -3$ imply $\alpha = 4$, $\beta = -5$, and $F(t) \in \Delta 5_1^\times$.

(4) We take $F(t) = t^2 - 3t + 3 - 3t^{-1} + t^{-2}$. Since $F(t) \equiv -2t + 2 - 2t^{-1} \pmod{t^2 - t + 1 - t^{-1} + t^{-2}}$, we have $p = -2$ and $q = 2$. There never exist α and β such that $\alpha = -2$, $\beta = 2$ by Lemma 11(2). We have $F(t) \notin \Delta 10_{132}^\times$. There never exist α and β such that $3\alpha + \beta = -4 \equiv 1 \pmod 5$, $3(-\alpha) - \beta = 4 \equiv -1 \pmod 5$ by Lemma 12(2). We have $F(t) \notin \Delta 5_1^\times$. The proof is complete. □

Problem. Find all pairs of α and β satisfying the conditions in Theorems 1 and 2.

6. Table

We apply the result in this paper for prime knots with ten crossings or less in the table of Rolfsen [7]. Here, 0 means the trivial knot, and N_n^* means the mirror image of N_n. Since 10_{161} and 10_{162} in [7] have the same knot type, here 10_n ($n \geqslant 162$) means 10_{n+1} in [7]. We use the following result by Murasugi [3]: *Let k be a knot,* k_1 *a knot obtained from k by a single crossing change. Then, we have* $|\sigma(k) - \sigma(k_1)| \leqslant 2$. We remark that $\sigma(10_{132}) = 0$, $\sigma(5_1) = 4$.

The symbol in each entry means the following: "a" means that k belongs to $10_{132}^{\times}$ (respectively $5_1^{\times}$). "b" means that $\Delta_k(t)$ belongs to $\Delta 10_{132}^{\times}$ (respectively $\Delta 5_1^{\times}$), but not "a". "c" means that $\Delta_k(t)$ belongs to $\Delta 10_{132}^{\times}$ (respectively $\Delta 5_1^{\times}$), but "a" or not "a" could not be decided. "d" means that $\Delta_k(t)$ does not belong to $\Delta 10_{132}^{\times}$ (respectively $\Delta 5_1^{\times}$). "e" means that "d" or not "d" could not be decided, but not "a". "?" means that there is no information.

	10_{132}	5_1		10_{132}	5_1		10_{132}	5_1
0	*a*	*d*	9_1	*b*	*d*	9_{33}	*c*	*d*
3_1	*d*	*a*	9_1^*	*b*	*d*	9_{33}^*	*c*	*d*
3_1^*	*d*	*b*	9_2	*c*	?	9_{34}	*c*	*d*
4_1	*c*	*d*	9_2^*	*c*	?	9_{34}^*	*c*	*d*
5_1	*b*	*a*	9_3	*d*	*d*	9_{35}	*d*	?
5_1^*	*b*	*b*	9_3^*	*d*	*d*	9_{35}^*	*d*	?
5_2	*d*	*a*	9_4	*d*	*d*	9_{36}	*d*	*b*
5_2^*	*d*	*b*	9_4^*	*d*	*d*	9_{36}^*	*d*	*c*
6_1	*c*	*d*	9_5	*d*	?	9_{37}	*c*	*e*
6_1^*	*c*	*d*	9_5^*	*d*	?	9_{37}^*	*c*	*e*
6_2	*d*	*d*	9_6	*d*	*a*	9_{38}	*d*	*c*
6_2^*	*d*	*d*	9_6^*	*d*	*b*	9_{38}^*	*d*	*b*
6_3	*d*	*d*	9_7	*d*	*d*	9_{39}	*c*	*e*
7_1	*d*	*a*	9_7^*	*d*	*d*	9_{39}^*	*c*	?
7_1^*	*d*	*b*	9_8	*d*	*d*	9_{40}	*d*	*c*
7_2	*a*	*d*	9_8^*	*d*	*d*	9_{40}^*	*d*	*b*
7_2^*	*c*	*d*	9_9	*b*	*d*	9_{41}	*c*	*d*
7_3	*d*	*b*	9_9^*	*b*	*d*	9_{41}^*	*c*	*d*
7_3^*	*d*	*a*	9_{10}	*d*	*b*	9_{42}	*d*	*b*
7_4	*c*	*e*	9_{10}^*	*d*	*c*	9_{42}^*	*d*	*c*
7_4^*	*c*	*e*	9_{11}	*d*	*b*	9_{43}	*d*	*b*
7_5	*d*	*a*	9_{11}^*	*d*	*a*	9_{43}^*	*d*	*c*
7_5^*	*d*	*b*	9_{12}	*d*	*c*	9_{44}	*d*	*d*
7_6	*d*	*d*	9_{12}^*	*d*	*b*	9_{44}^*	*d*	*d*
7_6^*	*d*	*d*	9_{13}	*d*	*b*	9_{45}	*d*	*c*
7_7	*c*	*d*	9_{13}^*	*d*	*a*	9_{45}^*	*d*	*b*
7_7^*	*c*	*d*	9_{14}	*d*	*e*	9_{46}	*c*	*d*
8_1	*d*	*e*	9_{14}^*	*d*	*e*	9_{46}^*	*c*	*d*
8_1^*	*d*	*e*	9_{15}	*c*	*d*	9_{47}	*d*	*b*
8_2	*d*	*a*	9_{15}^*	*c*	*d*	9_{47}^*	*d*	*c*
8_2^*	*d*	*b*	9_{16}	*b*	*d*	9_{48}	*d*	*e*
8_3	*d*	*d*	9_{16}^*	*b*	*d*	9_{48}^*	*d*	?
8_4	*d*	*d*	9_{17}	*d*	*d*	9_{49}	*d*	*b*
8_4^*	*d*	*d*	9_{17}^*	*d*	*d*	9_{49}^*	*d*	*c*
8_5	*b*	*d*	9_{18}	*d*	*d*	10_1	*d*	*d*
8_5^*	*b*	*d*	9_{18}^*	*d*	*d*	10_1^*	*d*	*d*
8_6	*d*	*c*	9_{19}	*c*	*d*	10_2	*d*	*a*
8_6^*	*d*	*b*	9_{19}^*	*c*	*d*	10_2^*	*d*	*b*
8_7	*d*	*b*	9_{20}	*b*	*d*	10_3	*c*	*d*
8_7^*	*d*	*a*	9_{20}^*	*b*	*d*	10_3^*	*c*	*d*
8_8	*c*	*e*	9_{21}	*d*	*e*	10_4	*d*	*c*
8_8^*	*c*	*e*	9_{21}^*	*d*	?	10_4^*	*d*	*b*
8_9	*c*	*d*	9_{22}	*d*	*b*	10_5	*d*	*b*
8_{10}	*d*	*b*	9_{22}^*	*d*	*c*	10_5^*	*d*	*a*
8_{10}^*	*d*	*a*	9_{23}	*d*	*a*	10_6	*d*	*a*
8_{11}	*d*	*c*	9_{23}^*	*d*	*b*	10_6^*	*d*	*b*
8_{11}^*	*d*	*b*	9_{24}	*c*	*e*	10_7	*d*	*c*
8_{12}	*d*	*d*	9_{24}^*	*c*	*e*	10_7^*	*d*	*b*
8_{13}	*c*	*d*	9_{25}	*d*	*c*	10_8	*d*	*d*
8_{13}^*	*c*	*d*	9_{25}^*	*d*	*b*	10_8^*	*d*	*d*
8_{14}	*d*	*d*	9_{26}	*d*	*b*	10_9	*c*	*d*
8_{14}^*	*d*	*d*	9_{26}^*	*d*	*a*	10_9^*	*c*	*d*
8_{15}	*d*	*a*	9_{27}	*c*	*d*	10_{10}	?	*e*
8_{15}^*	*d*	*b*	9_{27}^*	*c*	*d*	10_{10}^*	?	*e*
8_{16}	*d*	*d*	9_{28}	*d*	*d*	10_{11}	*d*	*c*
8_{16}^*	*d*	*d*	9_{28}^*	*d*	*d*	10_{11}^*	*d*	*b*
8_{17}	*d*	*d*	9_{29}	*d*	*d*	10_{12}	*d*	*b*
8_{18}	*c*	*e*	9_{29}^*	*d*	*d*	10_{12}^*	*d*	*a*
8_{19}	*d*	*b*	9_{30}	*d*	*e*	10_{13}	*d*	*e*
8_{19}^*	*d*	*a*	9_{30}^*	*d*	*e*	10_{13}^*	*d*	*e*
8_{20}	*a*	*d*	9_{31}	*d*	*a*	10_{14}	*d*	*a*
8_{20}^*	*c*	*d*	9_{31}^*	*d*	*b*	10_{14}^*	*d*	*b*
8_{21}	*d*	*c*	9_{32}	*d*	*d*	10_{15}	*d*	*b*
8_{21}^*	*d*	*b*	9_{32}^*	*d*	*d*	10_{15}^*	*d*	*a*

(*continued on next page*)

	10_{132}	5_1		10_{132}	5_1		10_{132}	5_1
10_{16}	*d*	*b*	10_{54}	*d*	*b*	10_{91}	*d*	*d*
10^*_{16}	*d*	*c*	10^*_{54}	*d*	*c*	10^*_{91}	*d*	*d*
10_{17}	*c*	*d*	10_{55}	*d*	*d*	10_{92}	*b*	*d*
10_{18}	*d*	*c*	10^*_{55}	*d*	*d*	10^*_{92}	*b*	*d*
10^*_{18}	*d*	*b*	10_{56}	*b*	*e*	10_{93}	*d*	*c*
10_{19}	*d*	*d*	10^*_{56}	*b*	?	10^*_{93}	*d*	*b*
10^*_{19}	*d*	*d*	10_{57}	*c*	*e*	10_{94}	*d*	*d*
10_{20}	*d*	?	10^*_{57}	*c*	?	10^*_{94}	*d*	*d*
10^*_{20}	*d*	*e*	10_{58}	?	*e*	10_{95}	*d*	*d*
10_{21}	*b*	*c*	10^*_{58}	?	*e*	10^*_{95}	*d*	*d*
10^*_{21}	*b*	*b*	10_{59}	*d*	*b*	10_{96}	*d*	*e*
10_{22}	*c*	*d*	10^*_{59}	*d*	*c*	10^*_{96}	*d*	*e*
10^*_{22}	*c*	*d*	10_{60}	?	*e*	10_{97}	*d*	*e*
10_{23}	*d*	*d*	10^*_{60}	?	*e*	10^*_{97}	*d*	?
10^*_{23}	*d*	*d*	10_{61}	*d*	*b*	10_{98}	*b*	*d*
10_{24}	*d*	*c*	10^*_{61}	*d*	*c*	10^*_{98}	*b*	*d*
10^*_{24}	*d*	*b*	10_{62}	*b*	*b*	10_{99}	*c*	*d*
10_{25}	*b*	?	10^*_{62}	*b*	*c*	10_{100}	*b*	*c*
10^*_{25}	*b*	*e*	10_{63}	*d*	*c*	10^*_{100}	*b*	*b*
10_{26}	*c*	*d*	10^*_{63}	*d*	*b*	10_{101}	*d*	*e*
10^*_{26}	*c*	*d*	10_{64}	*d*	*d*	10^*_{101}	*d*	?
10_{27}	*d*	*d*	10^*_{64}	*d*	*d*	10_{102}	*d*	*e*
10^*_{27}	*d*	*d*	10_{65}	*d*	*b*	10^*_{102}	*d*	*e*
10_{28}	*d*	*e*	10^*_{65}	*d*	*c*	10_{103}	*d*	*c*
10^*_{28}	*d*	*e*	10_{66}	*b*	?	10^*_{103}	*d*	*b*
10_{29}	*d*	*c*	10^*_{66}	*b*	*e*	10_{104}	*d*	*d*
10^*_{29}	*d*	*b*	10_{67}	*d*	*c*	10^*_{104}	*d*	*d*
10_{30}	*d*	*c*	10^*_{67}	*d*	*b*	10_{105}	*d*	*d*
10^*_{30}	*d*	*b*	10_{68}	*d*	*e*	10^*_{105}	*d*	*d*
10_{31}	*d*	*e*	10^*_{68}	*d*	*e*	10_{106}	*d*	*b*
10^*_{31}	*d*	*e*	10_{69}	*d*	*b*	10^*_{106}	*d*	*c*
10_{32}	*c*	*d*	10^*_{69}	*d*	*c*	10_{107}	*d*	*e*
10^*_{32}	*c*	*d*	10_{70}	*d*	*b*	10^*_{107}	*d*	*e*
10_{33}	?	*e*	10^*_{70}	*d*	*c*	10_{108}	*d*	*b*
10_{34}	*d*	*e*	10_{71}	*d*	*e*	10^*_{108}	*d*	*c*
10^*_{34}	*d*	*e*	10^*_{71}	*d*	*e*	10_{109}	*c*	*e*
10_{35}	*c*	*d*	10_{72}	*d*	*b*	10_{110}	*d*	*c*
10^*_{35}	*c*	*d*	10^*_{72}	*d*	*c*	10^*_{110}	*d*	*b*
10_{36}	*d*	*d*	10_{73}	*d*	*c*	10_{111}	*d*	?
10^*_{36}	*d*	*d*	10^*_{73}	*d*	*b*	10^*_{111}	*d*	?
10_{37}	*d*	*e*	10_{74}	*d*	*c*	10_{112}	*d*	*c*
10_{38}	*d*	*d*	10^*_{74}	*d*	*b*	10^*_{112}	*d*	*b*
10^*_{38}	*d*	*d*	10_{75}	*c*	*d*	10_{113}	*d*	*d*
10_{39}	*b*	*d*	10^*_{75}	*c*	*d*	10^*_{113}	*d*	*d*
10^*_{39}	*b*	*d*	10_{76}	*d*	*d*	10_{114}	*d*	*e*
10_{40}	*d*	*b*	10^*_{76}	*d*	*d*	10^*_{114}	*d*	*e*
10^*_{40}	*d*	*c*	10_{77}	*d*	*b*	10_{115}	*c*	*d*
10_{41}	*d*	*d*	10^*_{77}	*d*	*c*	10_{116}	*d*	*c*
10^*_{41}	*d*	*d*	10_{78}	*b*	*d*	10^*_{116}	*d*	*b*
10_{42}	*c*	*d*	10^*_{78}	*b*	*d*	10_{117}	*d*	*b*
10^*_{42}	*c*	*d*	10_{79}	*c*	*d*	10^*_{117}	*d*	*c*
10_{43}	*d*	*e*	10_{80}	*b*	*d*	10_{118}	*d*	*d*
10_{44}	*d*	*d*	10^*_{80}	*b*	*d*	10_{119}	*c*	*d*
10^*_{44}	*d*	*d*	10_{81}	*c*	*e*	10^*_{119}	*c*	*d*
10_{45}	*c*	*d*	10_{82}	*d*	*c*	10_{120}	*d*	*c*
10_{46}	*b*	*d*	10^*_{82}	*d*	*b*	10^*_{120}	*d*	*b*
10^*_{46}	*b*	*d*	10_{83}	*d*	*b*	10_{121}	*d*	?
10_{47}	*b*	*d*	10^*_{83}	*d*	*c*	10^*_{121}	*d*	*e*
10^*_{47}	*b*	*d*	10_{84}	*d*	*b*	10_{122}	*c*	*e*
10_{48}	*c*	*d*	10^*_{84}	*d*	*c*	10^*_{122}	*c*	*e*
10^*_{48}	*c*	*d*	10_{85}	*d*	*c*	10_{123}	*c*	*d*
10_{49}	*b*	*d*	10^*_{85}	*d*	*b*	10_{124}	*b*	*d*
10^*_{49}	*b*	*d*	10_{86}	?	*e*	10^*_{124}	*b*	*d*
10_{50}	*d*	*d*	10^*_{86}	?	*e*	10_{125}	*c*	*d*
10^*_{50}	*d*	*d*	10_{87}	*c*	*d*	10^*_{125}	*c*	*d*
10_{51}	*d*	*e*	10^*_{87}	*c*	*d*	10_{126}	*c*	*d*
10^*_{51}	*d*	?	10_{88}	*c*	*d*	10^*_{126}	*c*	*d*
10_{52}	*d*	*d*	10_{89}	*d*	*d*	10_{127}	*b*	*d*
10^*_{52}	*d*	*d*	10^*_{89}	*d*	*d*	10^*_{127}	*b*	*d*
10_{53}	*d*	*c*	10_{90}	*d*	*e*	10_{128}	*d*	*d*
10^*_{53}	*d*	*b*	10^*_{90}	*d*	*e*	10^*_{128}	*d*	*d*

	10_{132}	5_1		10_{132}	5_1		10_{132}	5_1
10_{129}	*c*	*e*	10^*_{141}	*c*	*d*	10_{154}	*d*	*b*
10^*_{129}	*c*	*e*	10_{142}	*d*	*b*	10^*_{154}	*d*	*c*
10_{130}	*d*	*b*	10^*_{142}	*d*	*c*	10_{155}	*c*	*d*
10^*_{130}	*d*	*b*	10_{143}	*d*	*c*	10^*_{155}	*c*	*d*
10_{131}	*d*	*d*	10^*_{143}	*d*	*b*	10_{156}	*d*	*c*
10^*_{131}	*d*	*d*	10_{144}	*d*	*d*	10^*_{156}	*d*	*b*
10_{132}	*a*	*b*	10^*_{144}	*d*	*d*	10_{157}	*d*	*d*
10^*_{132}	*c*	*b*	10_{145}	*d*	*d*	10^*_{157}	*d*	*d*
10_{133}	*d*	*d*	10^*_{145}	*d*	*d*	10_{158}	*c*	*d*
10^*_{133}	*d*	*d*	10_{146}	*d*	*e*	10^*_{158}	*c*	*d*
10_{134}	*d*	*b*	10^*_{146}	*d*	*e*	10_{159}	*d*	*d*
10^*_{134}	*d*	*a*	10_{147}	*d*	*b*	10^*_{159}	*d*	*d*
10_{135}	*d*	*e*	10^*_{147}	*d*	*c*	10_{160}	*b*	*d*
10^*_{135}	*d*	*e*	10_{148}	*c*	*d*	10^*_{160}	*b*	*d*
10_{136}	*d*	*b*	10^*_{148}	*c*	*d*	10_{161}	*d*	*c*
10^*_{136}	*d*	*c*	10_{149}	*b*	*d*	10^*_{161}	*d*	*b*
10_{137}	*c*	*d*	10^*_{149}	*b*	*d*	10_{162}	*d*	?
10^*_{137}	*c*	*d*	10_{150}	*b*	*d*	10^*_{162}	*d*	*e*
10_{138}	*d*	*b*	10^*_{150}	*b*	*d*	10_{163}	*d*	*d*
10^*_{138}	*d*	*c*	10_{151}	*d*	*b*	10^*_{163}	*d*	*d*
10_{139}	*d*	*d*	10^*_{151}	*d*	*c*	10_{164}	?	*e*
10^*_{139}	*d*	*d*	10_{152}	*b*	*d*	10^*_{164}	?	*e*
10_{140}	*c*	*d*	10^*_{152}	*b*	*d*	10_{165}	*c*	*d*
10^*_{140}	*c*	*d*	10_{153}	*c*	*d*	10^*_{165}	*c*	*d*
10_{141}	*c*	*d*	10^*_{153}	*c*	*d*			

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